3.2 Properties of Determinants

Theorem 3. Row Operations

Let A be a square matrix.

1. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
2. If two rows of A are interchanged to produce B, then $\det B = -\det A$.
3. If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

Example 1. Each equation illustrates a property of determinants. State the property.

1. $$\underset{A}{\begin{vmatrix} 3 & -6 & 9 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix}} = 3 \underset{B}{\begin{vmatrix} 1 & -2 & 3 \\ 3 & 5 & -5 \\ 1 & 3 & 3 \end{vmatrix}}$$

Thm 3 (3).

3×R1 of B produces A, det A = 3det B.

2. $$\underset{A}{\begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 2 & 7 & 4 \end{vmatrix}} = \underset{B}{\begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & -4 \\ 0 & 3 & 0 \end{vmatrix}}$$

2×R1+R3 of B produces A. (Thm 3(1))

So det A = det (B)

Question: If A is 3×3, what is det (4A) in terms of det A? 4^3 det A

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to \begin{bmatrix} 4a & 4b & 4c \\ d & e & f \\ g & h & i \end{bmatrix} \to \begin{bmatrix} 4a & 4b & 4c \\ 4d & 4e & 4f \\ g & h & i \end{bmatrix} \to \begin{bmatrix} 4a & 4b & 4c \\ 4d & 4e & 4f \\ 4g & 4h & 4i \end{bmatrix} = 4A$$

Example 2. Let

$$\underset{A}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = 5,$$

find the determinate of $\begin{vmatrix} a & b & c \\ g & h & i \\ 3d+a & 3e+b & 3f+c \end{vmatrix}$ = −15

ANS

① R2 ⟷ R3
② R3 → 3 R3
③ R3 → R3 + R1

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \longrightarrow \begin{vmatrix} a & b & c \\ g & h & i \\ 3d+a & 3e+b & 3f+c \end{vmatrix}.$$

$$\det(A) = 5 \xrightarrow{\text{step ①}} -5 \xrightarrow{\text{step ②}} 3\times(-5) = -15 \xrightarrow{\text{step ③}} -15$$

Remark. Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges. If there are r interchanges, then

$$\det A = (-1)^r \det U$$

Notice that $\det U = u_{11} \cdot u_{22} \cdots u_{nn}$, which is the product of the diagonal entries of U. If A is invertible, the entries u_{ii} are all pivots. Otherwise, at least u_{nn} is zero. Thus

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

Theorem 4. A square matrix A is invertible if and only if $\det A \neq 0$.

Example 3. Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

ANS: $A = \begin{bmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{bmatrix}$

We know A is invertible $\Leftrightarrow \det A \neq 0$?

$\Leftrightarrow$ The columns of A form a linearly independent set.

(The invertible Matrix Theorem)

We compute

$$\det A = 7 \cdot (-1)^{1+3} \cdot \begin{vmatrix} -4 & 5 \\ -6 & 7 \end{vmatrix} - 5 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 7 & -8 \\ -4 & 5 \end{vmatrix}$$

(expansion across the 3rd column)

$$= 7 \cdot (-28+30) - 5 \cdot (35-32)$$

$$= 14 - 15 = -1 \neq 0$$

Thus $\det A \neq 0$. So the given vectors are linearly independent.

Column Operations

Theorem 5. If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Determinants and Matrix Products

Theorem 6 Multiplicative Property
If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Example 4. Compute $\det A^4$, where $A = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 4 \\ 1 & 2 & 1 \end{bmatrix}$.

ANS: $\det A^4 = \det(A \cdot A \cdot A \cdot A) \overset{\text{Thm 6}}{=\!=} (\det A)^4$

$$\det A = 2 \cdot \begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 2 \cdot (2-8) + 2 \cdot (4-2)$$

$$= -12 + 4$$

$$= -8$$

Thus $\det A^4 = (\det A)^4 = 4096$

Exercise 5. Use determinants to find out if the matrix is invertible.

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{bmatrix}$$

ANS: $\det(A) = 2 \cdot \begin{vmatrix} 3 & 2 \\ 9 & 2 \end{vmatrix} - 6 \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}$

$$= 2 \cdot (6-18) - 6(2-6)$$

$$= 2 \cdot (-12) - 6(-4)$$

$$= -24 + 24 = 0.$$ So $\det(A) = 0$.

By Thm 4, A is not invertible.